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On the implementation of 'pseudo-harmonic' extension

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The 'pseudo-harmonic' extension is an approximation to the common harmonic extension for extending a function over a domain based on its trace along the boundary of the domain. On a circle the two extension methods produce identical results. We present explicit formulas for the computation of distance functions and intersection points needed in the 'pseudo-harmonic' extension on a circle, a square, and a pentagon. While the harmonic extension needs the solution of a Laplace problem for each new boundary function, the 'pseudo-harmonic' extension can reuse the distance functions and intersection points for any piecewise continuous function defined on the boundary of the domain.

1 Introduction

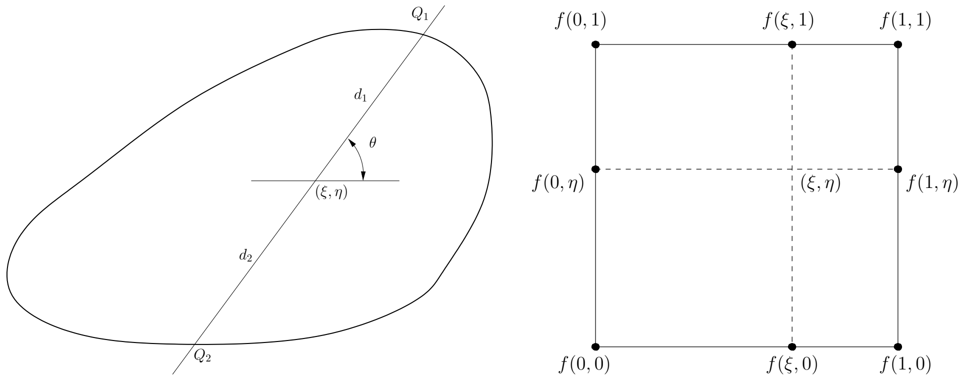
Extension of a function over a domain based on its trace along the boundary of the domain is a well studied problem. For a general domain $\Omega \in \mathbb{R}^d$, $d = 2, 3$, with f defined on $\partial\Omega$, a common method to find u over Ω such that $u_{|\partial\Omega} = f_{|\partial\Omega}$ is to solve the Laplace problem: Find u such that

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega, \\ u &= f \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

This method is often referred to as the harmonic extension, and it is very robust with respect to different domains Ω. Since the harmonic extension requires the solution of a Laplace problem, we are interested in a more explicit extension method.

For convex domains with piecewise differentiable boundaries, Gordon and Wixom introduced 'pseudo-harmonic' extension in [4]. On a bounded and convex domain $\Omega \subset \mathbb{R}^2$ where a function f is given on the boundary, the extension u of f to the domain Ω is defined as

$$u(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{d_2(\theta)}{d_1(\theta)+d_2(\theta)} f(Q_1(\theta)) + \frac{d_1(\theta)}{d_1(\theta)+d_2(\theta)} f(Q_2(\theta)) \right] d\theta, \tag{1.2}$$



(a) 'Pseudo-harmonic' extension on a bounded convex domain (taken from [4]).

(b) Transfinite extension on the unit square.

Figure 1: Boundary points of influence on an arbitrary point (ξ, η).

where Q_1 and Q_2 are the intersections between $\partial\Omega$ and the line through the point (ξ, η) at inclination θ, and d_1 and d_2 are the distances from (ξ, η) to these intersection points; see Figure 1(a) (taken from [4]). We use (ξ, η) as reference coordinates, since 'pseudo-harmonic' extension, and other methods of extension, is often used to map a reference domain to a generic domain, giving $(x(\xi, \eta), y(\xi, \eta)$. If the domain Ω is a circle, and f is piecewise continuous on its boundary, it is shown in [4] that the 'pseudo-harmonic' extension is equivalent to the harmonic extension found as the solution of the Laplace problem (1.1). For non-circular domains, the 'pseudo-harmonic' extension is an approximation to the harmonic extension found on the given domain.

The 'pseudo-harmonic' extension is well-defined both on domains with smooth boundaries, and on domains whose boundaries have corners. For comparison, the transfinite extension method introduced by Gordon and Hall in [2, 3], only applies to curvilinear quadrilateral domains. At each point (ξ, η), the 'pseudo-harmonic' extension $u(\xi, \eta)$ defined in (1.2) depends on the value of f along the *entire* boundary of Ω, while the extension defined through transfinite extension only depends on the eight boundary points indicated in Figure 1(b).

The main difficulty in using the 'pseudo-harmonic' extension is the computation of the intersection points Q_1 and Q_2. For general domains this involves finding the roots of a nonlinear equation. For simple domains however, the intersection points may be found analytically, and in Section 2 we will show this for a circle. In Sections 3 and 4 we present the same procedure for a square and a pentagon, both with sides of length one.

The extension u in (1.2) is equivalent to

$$u(\xi, \eta) = \frac{1}{\pi} \int_0^{2\pi} \frac{d_2(\theta)}{d_1(\theta) + d_2(\theta)} f(Q_1(\theta)) d\theta, \tag{1.3}$$

and following the discussion in [4] we evaluate the integral in (1.3) by a trapezoidal integration rule. For n equally spaced points $\theta_i = 2\pi i/n + \bar{\theta}$, where $\bar{\theta}$ is an additive constant, we get

$$u(\xi, \eta) \approx \frac{2}{n} \sum_{i=0}^{n-1} (\frac{d_2(\theta_i)}{d_1(\theta_i) + d_2(\theta_i)}) f(Q_1(\theta_i)). \tag{1.4}$$

At each point (ξ, η) in Ω we thus have to find the distance functions $d_1(\theta_i)$ and $d_2(\theta_i)$ and evaluate the boundary function at the intersection point $Q_1(\theta_i)$ for n different values of θ_i. We remark that the distance functions and the intersection points are independent of the

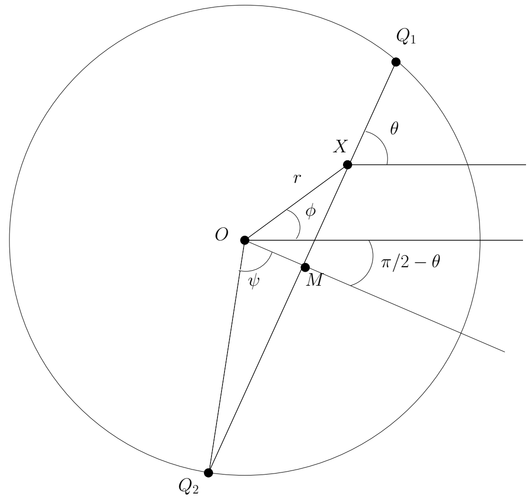


Figure 2: Circle with radius one.

boundary function f, and for a different choice of f the distance functions and boundary points are reused.

In [5], the 'pseudo-harmonic' extension is compared with several other extension methods: the harmonic extension, a generalized version of the transfinite extension, and the mean value extension [1]. A comparison is done regarding the ability to map a computational grid from a reference domain, e.g. a circle, a square, or a pentagon, to a generic, but topologically similar domain. To map a computational grid from one domain to another, the extensions $x(\xi,\eta)$ and $y(\xi,\eta)$ are computed separately by evaluating for example the integral in (1.3) at each point (ξ,η) in the computational grid that is defined on the reference domain.

2 Intersection points and distance functions on a circle

To find the intersection points and distance functions on the unit circle, we let (r,ϕ) denote the polar coordinates of a given point X in the interior of the circle; see Figure 2. The intersection point $Q_1(\theta)$ corresponding to the point X is defined by the polar coordinates $(1,\omega(\theta))$, where $\omega(\theta)$ is the angle of $Q_1(\theta)$ relative to the center of the circle. To find $\omega(\theta)$, we consider the triangle formed by OXQ_1, where O is the center of the circle. The angle between XO and XQ_1 is $\phi+\pi-\theta$. Basic trigonometry gives

$$\omega(\theta;r,\phi)=\theta-\sin^{-1}(r\sin(\phi+\pi-\theta)), \tag{2.1}$$

where $(\theta;r,\phi)$ indicates that for a given θ this is the angle corresponding to the point $X=(r,\phi)$.

To find the distance functions d_1 and d_2 we let M denote the midpoint of the line between Q_1 and Q_2; see Figure 2. The angle between the horizontal and the normal from the center of the circle to this line is $\pi/2-\theta$. Thus the distance from M to X is

$$|MX|=r\cos(\phi-\theta). \tag{2.2}$$

We let ψ denote the angle between the lines OQ_2 and OM. Since the radius of the circle is one, the distance from M to Q_2 is given by

$$|MQ_2|=\sin(\psi). \tag{2.3}$$

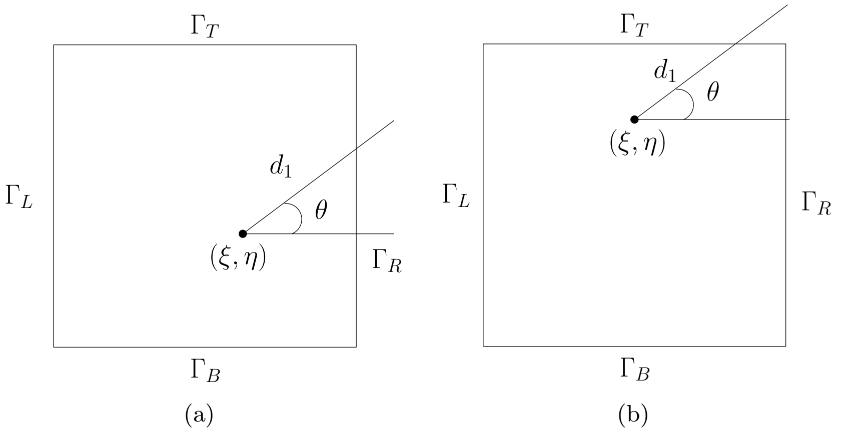


Figure 3: The intersected boundary line depends on both θ and (ξ, η).

One way to find ψ is to use that M is the midpoint of the line between Q_1 and Q_2. The two triangles OMQ_1 and OMQ_2 are equal in shape, and we have $\psi = \omega(\theta) + \pi/2 - \theta$. We insert the expression for $\omega(\theta)$ found in (2.1), and after some rewriting we get

$$\psi = \cos^{-1}(r \sin(\phi + \pi - \theta)). \tag{2.4}$$

Finally $d_1 = |MQ_2| - |MX|$ and $d_2 = |MQ_2| + |MX|$ gives

$$d_1(\theta; r, \phi) = \sin(\cos^{-1}(r \sin(\phi + \pi - \theta))) - r \cos(\phi - \theta) \tag{2.5}$$

$$d_2(\theta; r, \phi) = \sin(\cos^{-1}(r \sin(\phi + \pi - \theta))) + r \cos(\phi - \theta). \tag{2.6}$$

For a given boundary function $f(\omega)$, we insert (2.1), (2.5), and (2.6) in the trapezoidal integration rule presented in (1.4) to find the extension of $f(\omega)$ to all points in the circle.

3 Intersection points and distance functions on a square

We now focus on the unit square, $\Omega = (0,1)^2$, and denote its boundary lines by

$$\begin{array}{lll} \Gamma_L: & \xi = 0, & \eta \in [0,1], \\ \Gamma_R: & \xi = 1, & \eta \in [0,1], \\ \Gamma_B: & \xi \in [0,1], & \eta = 0, \\ \Gamma_T: & \xi \in [0,1], & \eta = 1. \end{array} \tag{3.1}$$

For a given point $(\xi, \eta) \in \Omega$ and a given angle $\theta \in [0, 2\pi)$, the distance functions $d_1(\theta; \xi, \eta)$ and $d_2(\theta; \xi, \eta)$ are found by determining which of the boundary lines are intersected. If $\theta = 0$ there is no doubt which boundary lines are intersected, and we obviously have $d_1 = 1 - \xi$ and $d_2 = \xi$. Similar expressions are also found for $\theta = \pi/2, \pi$, and $3\pi/2$.

For other angles the boundary lines intersected will depend on the point (ξ, η) as well. To illustrate this we consider $\theta \in (0, \pi/2)$. In Figures 3(a) and 3(b) we see that different boundary lines are intersected for different choices of (ξ, η). In Figure 3(a) we intersect Γ_R, and get the distance function

$$d_1 = \frac{1 - \xi}{\cos(\theta)}, \tag{3.2}$$

while we in Figure 3(b) intersect Γ_T, and get the distance function

$$d_1 = \frac{1 - \eta}{\sin(\theta)}. \tag{3.3}$$

k	$d_1(\theta;\xi,\eta)$	$d_2(\theta;\xi,\eta)$
1	$\min\left\{\frac{1-\xi}{\cos(\theta)},\frac{1-\eta}{\sin(\theta)}\right\}$	$\min\left\{\frac{\xi}{\cos(\theta)},\frac{\eta}{\sin(\theta)}\right\}$
2	$\min\left\{\frac{\xi}{\cos(\pi-\theta)},\frac{1-\eta}{\sin(\theta)}\right\}$	$\min\left\{\frac{1-\xi}{\cos(\pi-\theta)},\frac{\eta}{\sin(\theta)}\right\}$
3	$\min\left\{\frac{\xi}{\cos(\theta-\pi)},\frac{\eta}{\sin(\theta-\pi)}\right\}$	$\min\left\{\frac{1-\xi}{\cos(\theta-\pi)},\frac{1-\eta}{\sin(\theta-\pi)}\right\}$
4	$\min\left\{\frac{1-\xi}{\cos(2\pi-\theta)},\frac{\eta}{\sin(2\pi-\theta)}\right\}$	$\min\left\{\frac{\xi}{\cos(2\pi-\theta)},\frac{1-\eta}{\sin(2\pi-\theta)}\right\}$

Table 1: The distance functions $d_1(\theta;\xi,\eta)$ and $d_2(\theta;\xi,\eta)$ on the unit square found for different choices of the inclination angle θ. The angle is given by $(k-1)\pi/2 < \theta < k\pi/2$.

To identify the correct distance function for any point in Ω, we simply let

$$d_1 = \min\left\{\frac{1-\xi}{\cos(\theta)},\frac{1-\eta}{\sin(\theta)}\right\}. \tag{3.4}$$

This expression holds for all $(\xi,\eta)\in\Omega$ and $\theta\in(0,\pi/2)$, since a line crossing either Γ_R or Γ_T will cross the extension of the other boundary line outside Ω. The minimum of these two distances will then be the distance to the boundary of Ω.

The distance function d_2 is found by tracing the line through (ξ,η) with inclination θ in the direction $\pi+\theta$, and for $\theta\in(0,\pi/2)$ it will intersect either Γ_L or Γ_B. Based on the same arguments as for d_1, the distance d_2 is given by

$$d_2 = \min\left\{\frac{\xi}{\sin(\theta)},\frac{\eta}{\cos(\theta)}\right\}. \tag{3.5}$$

A similar procedure is used to find the distance functions for $(k-1)\pi/2 < \theta < k\pi/2$, $k = 2, 3$, and 4. For $k = 2$ the line with inclination $\theta\in(\pi/2,\pi)$ will cross either Γ_T or Γ_L, for $k = 3$ the line with inclination $\theta\in(\pi,3\pi/2)$ will cross either Γ_L or Γ_B, and for $k = 4$ the line with inclination $\theta\in(3\pi/2,2\pi)$ will cross either Γ_B or Γ_R. The resulting distance functions are presented in Table 1. Once the distance functions are found, the interpolation points Q_1 and Q_2 are given by

$$\begin{aligned} Q_1 &= (\xi + d_1\cos(\theta), \eta + d_1\sin(\theta)),\\ Q_2 &= (\xi - d_2\cos(\theta), \eta - d_2\sin(\theta)). \end{aligned} \tag{3.6}$$

To illustrate, we show in Figure 4 how the 'pseudo-harmonic' extension extends a function f given by a parabolic profile on each side of the square $(0,1)^2$.

4 Intersection points and distance functions on a pentagon

On the uniform pentagon we proceed in the same way as for the unit square, but now the process is more involved. Depending on the choice of the inclination angle θ, the line from any given point $(\xi,\eta)\in\Omega$ (where Ω now is the uniform pentagon) has two or three boundary lines as candidates for intersection. To see this we use Figures 5(a) and 5(b). The boundary lines of Ω are denoted in a counterclockwise manner by $\Gamma_A,\Gamma_B,\Gamma_C,\Gamma_D$, and Γ_E, where Γ_A is the bottom horizontal line, and the length of each side is set to 1. Furthermore, the corner between Γ_A and Γ_E has coordinates $(0,0)$. For $0\le\theta<\pi/5$, only the boundary lines Γ_B and Γ_C are eligible for intersection by the line through a point (ξ,η) with inclination θ; see Figure 5(a). Which of the two lines is intersected, is determined

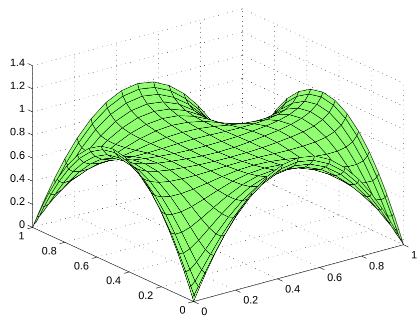


Figure 4: Extension of a parabolic profile.

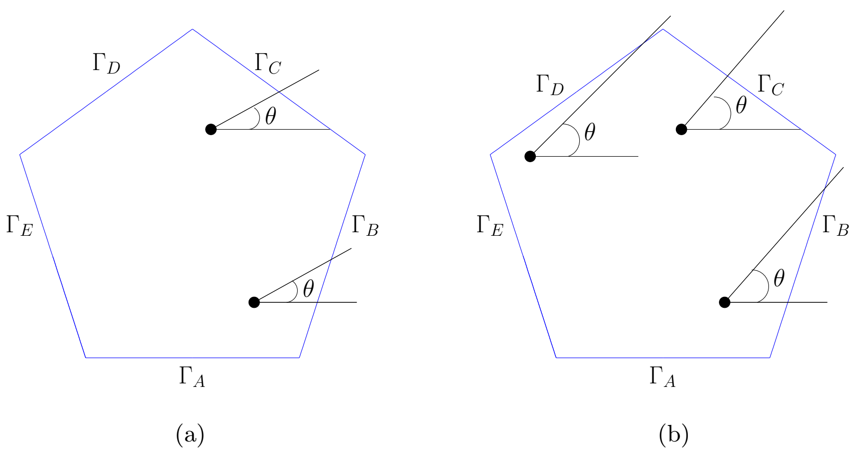


Figure 5: The number of possible boundary lines to intersect depends on θ. The intersected boundary line depends on both θ and (ξ, η).

by both θ and (ξ, η). For $\pi/5 \leq \theta < 2\pi/5$ both Γ_B, Γ_C, and Γ_D are possible intersection candidates; see Figure 5(b). Continuing in this fashion, we consider 10 different cases, given by

$$(k-1)\pi/5 \leq \theta < k\pi/5, \quad k = 1, ..., 10, \tag{4.1}$$

where $\theta = k\pi/5$ corresponds to lines parallel to one of the boundary lines.

The distance from any point (ξ, η) to the extended boundary line Γ_B is, for $\theta \in [0, 2\pi/5)$, given by

$$d_1 = \left(1 - \xi + \frac{\eta}{\tan(2\pi/5)}\right) \frac{\sin(3\pi/5)}{\sin(2\pi/5 - \theta)}. \tag{4.2}$$

Similarly, the distance to the extended boundary line Γ_C is, for $\theta \in [0, 4\pi/5)$, given by

$$d_1 = \left(\xi_t - \xi + \frac{\eta_t - \eta}{\tan(\pi/5)}\right) \frac{\sin(\pi/5)}{\sin(4\pi/5 - \theta)}, \tag{4.3}$$

where $(\xi_t, \eta_t) = (0.5, sin(2\pi/5) + sin(\pi/5))$ is the corner between Γ_C and Γ_D. The minimum of these two distances gives the correct distance corresponding to Figure 5(a), where $\theta \in [0, \pi/5)$. In Tables 2 and 3 we present the distance functions on the uniform pentagon for the different intervals of the inclination angle θ given in (4.1).

We recall that the distance functions are independent of the boundary function in (1.3). Once the distance functions are defined, the extension of any piecewise continuous boundary function is rapid.

As for the unit square, the interpolation points Q_1 and Q_2 are given by

$$\begin{aligned} Q_1 &= (\xi + d_1 \cos(\theta), \eta + d_1 \sin(\theta)), \\ Q_2 &= (\xi - d_2 \cos(\theta), \eta - d_2 \sin(\theta)). \end{aligned} \tag{4.4}$$

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k	$d_1(\theta;\xi,\eta)$
1	$\min\left\{\left(1-\xi+\frac{\eta}{\tan(2\pi/5)}\right)\frac{\sin(3\pi/5)}{\sin(2\pi/5-\theta)},\left(\xi_t-\xi+\frac{\eta_t-\eta}{\tan(\pi/5)}\right)\frac{\sin(\pi/5)}{\sin(4\pi/5-\theta)}\right\}$
2	$\min\left\{\left(1-\xi+\frac{\eta}{\tan(2\pi/5)}\right)\frac{\sin(3\pi/5)}{\sin(2\pi/5-\theta)},\left(\xi_t-\xi+\frac{\eta_t-\eta}{\tan(\pi/5)}\right)\frac{\sin(\pi/5)}{\sin(4\pi/5-\theta)},\left(\eta_t-\eta+\frac{\xi-\xi_t}{\tan(\pi/2-\pi/5)}\right)\frac{\sin(\pi/2+\pi/5)}{\sin(\theta-\pi/5)}\right\}$
3	$\min\left\{\left(\xi_t-\xi+\frac{\eta_t-\eta}{\tan(\pi/5)}\right)\frac{\sin(\pi/5)}{\sin(4\pi/5-\theta)},\left(\eta_t-\eta+\frac{\xi-\xi_t}{\tan(\pi/2-\pi/5)}\right)\frac{\sin(\pi/2+\pi/5)}{\sin(\theta-\pi/5)}\right\}$
4	$\min\left\{\left(\xi_t-\xi+\frac{\eta_t-\eta}{\tan(\pi/5)}\right)\frac{\sin(\pi/5)}{\sin(4\pi/5-\theta)},\left(\eta_t-\eta+\frac{\xi-\xi_t}{\tan(\pi/2-\pi/5)}\right)\frac{\sin(\pi/2+\pi/5)}{\sin(\theta-\pi/5)},\left(\xi+\frac{\eta}{\tan(2\pi/5)}\right)\frac{\sin(3\pi/5)}{\sin(\theta-3\pi/5)}\right\}$
5	$\min\left\{\left(\eta_t-\eta+\frac{\xi-\xi_t}{\tan(\pi/2-\pi/5)}\right)\frac{\sin(\pi/2+\pi/5)}{\sin(\theta-\pi/5)},\left(\xi+\frac{\eta}{\tan(2\pi/5)}\right)\frac{\sin(3\pi/5)}{\sin(\theta-3\pi/5)}\right\}$
6	$\min\left\{\left(\eta_t-\eta+\frac{\xi-\xi_t}{\tan(\pi/2-\pi/5)}\right)\frac{\sin(\pi/2+\pi/5)}{\sin(\theta-\pi/5)},\left(\xi+\frac{\eta}{\tan(2\pi/5)}\right)\frac{\sin(3\pi/5)}{\sin(\theta-3\pi/5)},\frac{\eta}{\sin(\theta-\pi)}\right\}$
7	$\min\left\{\left(\xi+\frac{\eta}{\tan(2\pi/5)}\right)\frac{\sin(3\pi/5)}{\sin(\theta-3\pi/5)},\ \frac{\eta}{\sin(\theta-\pi)}\right\}$
8	$\min\left\{\left(\xi+\frac{\eta}{\tan(2\pi/5)}\right)\frac{\sin(3\pi/5)}{\sin(\theta-3\pi/5)},\left(1-\xi+\frac{\eta}{\tan(2\pi/5)}\right)\frac{\sin(3\pi/5)}{\sin(\theta-7\pi/5)},\frac{\eta}{\sin(\theta-\pi)}\right\}$
9	$\min\left\{\left(1-\xi+\frac{\eta}{\tan(2\pi/5)}\right)\frac{\sin(3\pi/5)}{\sin(\theta-7\pi/5)},\ \frac{\eta}{\sin(\theta-\pi)}\right\}$
10	$\min\left\{\left(1-\xi+\frac{\eta}{\tan(2\pi/5)}\right)\frac{\sin(3\pi/5)}{\sin(\theta-7\pi/5)},\left(\xi_t-\xi+\frac{\eta_t-\eta}{\tan(\pi/5)}\right)\frac{\sin(\pi/5)}{\sin(\theta-9\pi/5)},\frac{\eta}{\sin(\theta-\pi)}\right\}$

Table 2: The distance function $d_1(\theta;\xi,\eta)$ on the uniform pentagon found for different intervals of the inclination angle θ. The inclination angle is given by $(k-1)\pi/5 \leq \theta < k\pi/5$.

k	$d_2(\theta;\xi,\eta)$
1	$\min\left\{\left(\xi+\frac{\eta}{\tan(2\pi/5)}\right)\frac{\sin(2\pi/5)}{\sin(3\pi/5-\theta)},\left(\eta_t-\eta+\frac{\xi-\xi_t}{\tan(\pi/2-\pi/5)}\right)\frac{\sin(\pi/2-\pi/5)}{\sin(\pi/5-\theta)},\frac{\eta}{\sin(\theta)}\right\}$
2	$\min\left\{\left(\xi+\frac{\eta}{\tan(2\pi/5)}\right)\frac{\sin(2\pi/5)}{\sin(3\pi/5-\theta)},\ \frac{\eta}{\sin(\theta)}\right\}$
3	$\min\left\{\left(\xi+\frac{\eta}{\tan(2\pi/5)}\right)\frac{\sin(2\pi/5)}{\sin(3\pi/5-\theta)},\left(\eta-\frac{\xi-1}{\tan(\pi/2-2\pi/5)}\right)\frac{\sin(2\pi/5+\pi/2)}{\sin(\theta-2\pi/5)},\frac{\eta}{\sin(\theta)}\right\}$
4	$\min\left\{\left(\eta-\frac{\xi-1}{\tan(\pi/2-2\pi/5)}\right)\frac{\sin(2\pi/5+\pi/2)}{\sin(\theta-2\pi/5)},\ \frac{\eta}{\sin(\theta)}\right\}$
5	$\min\left\{\left(\eta-\frac{\xi-1}{\tan(\pi/2-2\pi/5)}\right)\frac{\sin(2\pi/5+\pi/2)}{\sin(\theta-2\pi/5)},\left(\xi_t-\xi+\frac{\eta_t-\eta}{\tan(\pi/5)}\right)\frac{\sin(4\pi/5)}{\sin(\theta-4\pi/5)},\frac{\eta}{\sin(\theta)}\right\}$
6	$\min\left\{\left(\eta-\frac{\xi-1}{\tan(\pi/2-2\pi/5)}\right)\frac{\sin(2\pi/5+\pi/2)}{\sin(\theta-2\pi/5)},\left(\xi_t-\xi+\frac{\eta_t-\eta}{\tan(\pi/5)}\right)\frac{\sin(4\pi/5)}{\sin(\theta-4\pi/5)}\right\}$
7	$\min\left\{\left(\eta-\frac{\xi-1}{\tan(\pi/2-2\pi/5)}\right)\frac{\sin(2\pi/5+\pi/2)}{\sin(\theta-2\pi/5)},\left(\xi_t-\xi+\frac{\eta_t-\eta}{\tan(\pi/5)}\right)\frac{\sin(4\pi/5)}{\sin(\theta-4\pi/5)},\left(\eta_t-\eta+\frac{\xi-\xi_t}{\tan(\pi/2-\pi/5)}\right)\frac{\sin(\pi/2+\pi/5)}{\sin(\theta-6\pi/5)}\right\}$
8	$\min\left\{\left(\xi_t-\xi+\frac{\eta_t-\eta}{\tan(\pi/5)}\right)\frac{\sin(4\pi/5)}{\sin(\theta-4\pi/5)},\left(\eta_t-\eta+\frac{\xi-\xi_t}{\tan(\pi/2-\pi/5)}\right)\frac{\sin(\pi/2+\pi/5)}{\sin(\theta-6\pi/5)}\right\}$
9	$\min\left\{\left(\xi_t-\xi+\frac{\eta_t-\eta}{\tan(\pi/5)}\right)\frac{\sin(4\pi/5)}{\sin(\theta-4\pi/5)},\left(\eta_t-\eta+\frac{\xi-\xi_t}{\tan(\pi/2-\pi/5)}\right)\frac{\sin(\pi/2+\pi/5)}{\sin(\theta-6\pi/5)},\left(\xi+\frac{\eta}{\tan(2\pi/5)}\right)\frac{\sin(3\pi/5)}{\sin(\theta-8\pi/5)}\right\}$
10	$\min\left\{\left(\eta_t-\eta+\frac{\xi-\xi_t}{\tan(\pi/2-\pi/5)}\right)\frac{\sin(\pi/2+\pi/5)}{\sin(\theta-6\pi/5)},\left(\xi+\frac{\eta}{\tan(2\pi/5)}\right)\frac{\sin(3\pi/5)}{\sin(\theta-8\pi/5)}\right\}$

Table 3: The distance function $d_2(\theta;\xi,\eta)$ on the uniform pentagon found for different intervals of the inclination angle θ. The inclination angle is given by $(k-1)\pi/5 \leq \theta < k\pi/5$.