The Reduced Basis Element Method: Offline-Online Decomposition in the Nonconforming, Nonaffine Case

A. E. Løvgren*, Y. Maday†, and E. M. Rønquist‡

*Center for Biomedical Computing
Simula Research Laboratory

†Laboratoire J.-L. Lions
Université Pierre et Marie Curie-Paris6,
and Division of Applied Mathematics
Brown University

‡Department of Mathematical Sciences
Norwegian University of Science and Technology

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Outline

- Reduced basis ingredients
- Flow problems on complex geometries
- Domain decomposition
- The geometry as a parameter -> non-affine
- Reduced basis elements -> non-conforming
- Smoothing process
- Conforming error analysis
- Offline basis for the smoothing process

Reduced basis ingredients

- Parameter set: $S_N = \{\mu_i\}_{i=1}^N$.
- Affine decomposition: $A_N(\mu) = \sum_{q=1}^{Q} \beta^q(\mu) A_N^q$.
- Reduced basis solution: $u_N(\mu) = \sum_{i=1}^{N} \alpha_i(\mu) u_i$.
- Output of interest: $s_N(\mu) = \sum_{i=1}^{N} \alpha_i(\mu) s_i$.
- A posteriori error analysis: $s^-(\mu) \leq s_N(\mu) \leq s^+(\mu)$.

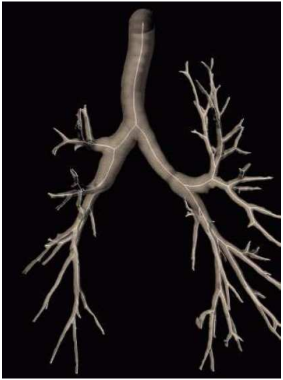
Flow problems on complex geometries

The steady Stokes equations:
find $\mathbf{u} \in X$ and $p \in M$

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}; \mu) + b(\mathbf{v}, p; \mu) &= l(\mathbf{u}; \mu) \quad \forall \mathbf{v} \in X \\ b(\mathbf{u}, q; \mu) &= 0 \quad \forall q \in M \end{aligned}$$

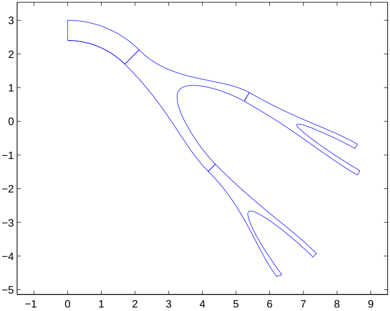
Output of interest:

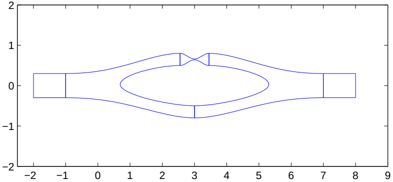
$$s(\mathbf{u}; \mu) = l(\mathbf{u}; \mu).$$



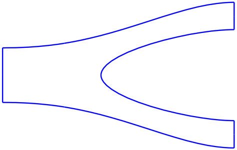
Domain decomposition

2D networks:

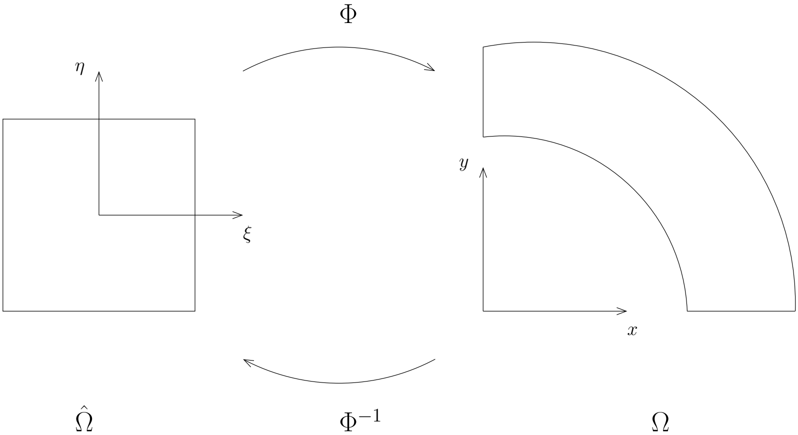




Building blocks:



The geometry as a parameter



The geometry as a parameter

The viscous operator in the Stokes problem:

$$a(\mathbf{u}, \mathbf{v}; \Phi) = \nu \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} d\Omega.$$

The viscous operator represented on the reference domain:

$$a(\mathbf{u}, \mathbf{v}; \Phi) = \nu \int_{\widehat{\Omega}} \mathcal{J}^{-T} \widehat{\nabla}(\mathbf{u} \circ \Phi) \cdot \mathcal{J}^{-T} \widehat{\nabla}(\mathbf{v} \circ \Phi) |J| d\widehat{\Omega},$$

where $\mathcal{J}$ is the Jacobian of Φ, and $\widehat{\nabla} = \mathcal{J}^T \nabla$.

The geometry as a parameter

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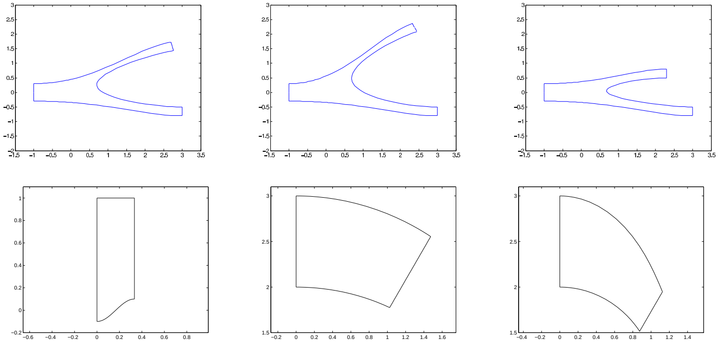
where $\mathcal{J}$ is the Jacobian of Φ, and $\widehat{\nabla} = \mathcal{J}^T \nabla$.

The divergence operator represented on the reference domain:

$$b(\mathbf{v}, p; \Phi) = -\int_{\widehat{\Omega}} (p \circ \Phi) \widehat{\nabla} \cdot [\mathcal{J}^{-1}(\mathbf{v} \circ \Phi)] |J| d\widehat{\Omega}.$$

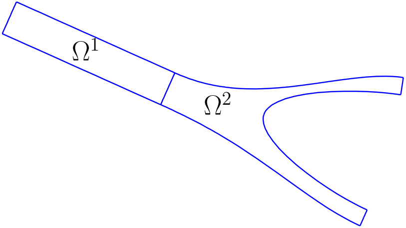
The geometry as a parameter



Parameter set: $\Phi_i : \widehat{\Omega} \longrightarrow \Omega_i, \quad i = 1, ..., N.$

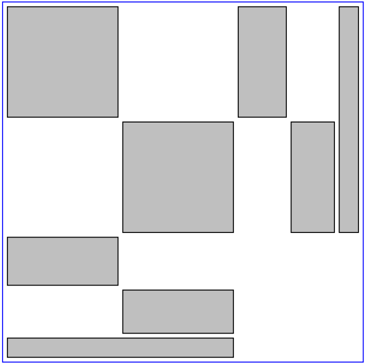
Corresponding basis functions: $(\mathbf{u}_i, p_i), \quad i = 1, ..., N$. Stored on the reference domain: $\hat{p}_i = p_i \circ \Phi_i^{-1}, \quad \hat{\mathbf{u}}_i = \Psi(\mathbf{u}_i, \Phi_i) = \mathcal{J}_i^{-1}(\mathbf{u}_i \circ \Phi_i)|J_i|.$

Reduced basis element solution



$$
\begin{array}{lcll}
a(\mathbf{u}_N^1, \mathbf{v}; \Phi^1) + b(\mathbf{v}, p_N^1; \Phi^1) & = & l(\mathbf{v}; \Phi^1) & \forall\, \mathbf{v} \in X_N(\Omega^1) \\
b(\mathbf{u}_N^1, q; \Phi^1) & = & 0 & \forall\, q \in M_N(\Omega^1) \\
a(\mathbf{u}_N^2, \mathbf{v}; \Phi^2) + b(\mathbf{v}, p_N^2; \Phi^2) & = & l(\mathbf{v}; \Phi^2) & \forall\, \mathbf{v} \in X_N(\Omega^2) \\
b(\mathbf{u}_N^2, q; \Phi^2) & = & 0 & \forall\, q \in M_N(\Omega^2) \\
\int_\Gamma (\mathbf{u}_N^1 - \mathbf{u}_N^2) \cdot \mathbf{n} \psi ds & = & 0 & \forall\, \psi \in W^n(\Gamma) \\
\int_\Gamma (\mathbf{u}_N^1 - \mathbf{u}_N^2) \cdot \mathbf{t} \psi ds & = & 0 & \forall\, \psi \in W^t(\Gamma)
\end{array}
$$

Reduced basis element solution



Velocity:

$$\mathbf{u}_N = \sum_{i=1}^{2N} \alpha_i^1 \tilde{\mathbf{u}}_i^1 + \sum_{i=1}^{2N} \alpha_i^2 \tilde{\mathbf{u}}_i^2,$$

where $\tilde{\mathbf{u}}_i^1 = \Psi^{-1}(\hat{\mathbf{u}}_i, \Phi^1)$.

Pressure:

$$p_N = \sum_{i=1}^{N} \beta_i^1 \tilde{p}_i^1 + \sum_{i=1}^{N} \beta_i^2 \tilde{p}_i^2,$$

where $\tilde{p}_i^1 = \hat{p}_i \circ \Phi^1$.

Nonaffine parameter dependence - offline computation

$$\begin{aligned} A_{ij} = a(\tilde{\mathbf{u}}_i, \tilde{\mathbf{u}}_j; \Phi) &= \nu \int_{\widehat{\Omega}} \mathcal{J}^{-T} \widehat{\nabla}(\tilde{\mathbf{u}}_i \circ \Phi) \cdot \mathcal{J}^{-T} \widehat{\nabla}(\tilde{\mathbf{u}}_j \circ \Phi) |J| d\widehat{\Omega} \\ &= \nu \sum_{q=1}^{Q} a^q(\hat{\mathbf{u}}_i, \hat{\mathbf{u}}_j, g^q(\Phi)) \end{aligned}$$

Nonaffine parameter dependence - offline computation

$$A_{ij} = a(\tilde{\mathbf{u}}_i, \tilde{\mathbf{u}}_j; \Phi) = \nu \int_{\widehat{\Omega}} \mathcal{J}^{-T} \widehat{\nabla}(\tilde{\mathbf{u}}_i \circ \Phi) \cdot \mathcal{J}^{-T} \widehat{\nabla}(\tilde{\mathbf{u}}_j \circ \Phi) |J| d\widehat{\Omega}$$
$$= \nu \sum_{q=1}^{Q} a^q(\hat{\mathbf{u}}_i, \hat{\mathbf{u}}_j, g^q(\Phi))$$

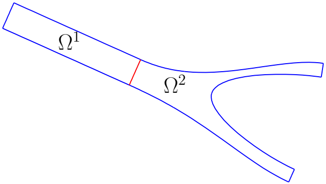
Using $\tilde{\mathbf{u}}_i = \Psi^{-1}(\hat{\mathbf{u}}_i, \Phi) = \frac{1}{|J|}\mathcal{J}(\hat{\mathbf{u}}_i \circ \Phi^{-1})$, we get

$$a(\tilde{\mathbf{u}}_i, \tilde{\mathbf{u}}_j; \Phi) = \nu \int_{\widehat{\Omega}} \mathcal{J}^{-T} \widehat{\nabla}(\frac{1}{|J|}\mathcal{J}\hat{\mathbf{u}}_i) \cdot \mathcal{J}^{-T} \widehat{\nabla}(\frac{1}{|J|}\mathcal{J}\hat{\mathbf{u}}_j) |J| d\widehat{\Omega},$$

and for $Q = 17$

$$a^1(\hat{\mathbf{u}}_i, \hat{\mathbf{u}}_j, g^1(\Phi)) = \int_{\widehat{\Omega}} \frac{1}{|J|^3} (\mathcal{J}_{11}^2 + \mathcal{J}_{21}^2)(\mathcal{J}_{12}^2 + \mathcal{J}_{22}^2) \left(\frac{\partial \hat{u}_{i\xi}}{\partial \xi} \frac{\partial \hat{u}_{j\xi}}{\partial \xi} + \frac{\partial \hat{u}_{i\eta}}{\partial \eta} \frac{\partial \hat{u}_{j\eta}}{\partial \eta} \right) d\widehat{\Omega}$$

Output bounds - nonconforming case



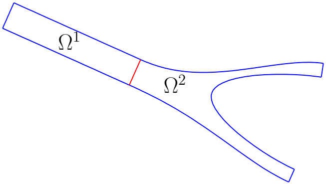
Since $\mathbf{u}_N \notin X(\Omega)$, we define the jump $[\mathbf{u}_e] = \mathbf{u}^l_{N_{|\Gamma_e}} - \mathbf{u}^k_{N_{|\Gamma_e}}$.

Find $\mathbf{u}^k_c \in X^{\Gamma_e}(\Omega^k)$ and $p^k_c \in M(\Omega^k)$, such that

$$
\begin{array}{lcll}
a(\mathbf{u}^k_c, \mathbf{v}; \Phi^k) + b(\mathbf{v}, p^k_c; \Phi^k) & = & 0 & \forall\, \mathbf{v} \in (H^1_0(\Omega^k))^2 \\
b(\mathbf{u}^k_c, q; \Phi^k) & = & 0 & \forall\, q \in M_N(\Omega^k),
\end{array}
$$

where $X^{\Gamma_e}(\Omega^k) = \{\mathbf{v} \in (H^1(\Omega^k))^2, \mathbf{v} = [\mathbf{u}_e] \text{ on } \Gamma_e, \mathbf{v} = 0 \text{ on } \partial\Omega^k \backslash \Gamma_e\}$.

Output bounds - nonconforming case



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Find $\mathbf{u}^k_c \in X^{\Gamma_e}(\Omega^k)$ and $p^k_c \in M(\Omega^k)$, such that

$$\begin{aligned} a(\mathbf{u}^k_c, \mathbf{v}; \Phi^k) + b(\mathbf{v}, p^k_c; \Phi^k) &= 0 \quad \forall\, \mathbf{v} \in (H^1_0(\Omega^k))^2 \\ b(\mathbf{u}^k_c, q; \Phi^k) &= 0 \quad \forall\, q \in M_N(\Omega^k), \end{aligned}$$

where $X^{\Gamma_e}(\Omega^k) = \{\mathbf{v} \in (H^1(\Omega^k))^2, \mathbf{v} = [\mathbf{u}_e] \text{ on } \Gamma_e, \mathbf{v} = 0 \text{ on } \partial\Omega^k \backslash \Gamma_e\}$.

The resulting $\tilde{\mathbf{u}}_N = \mathbf{u}_N + \sum_k \mathbf{u}^k_c$ is in $X(\Omega)$.

Output bounds - nonconforming case

Using conforming analysis, we get the output bounds

$$s^-(\tilde{\mathbf{u}}_N;\Phi) = 2l(\tilde{\mathbf{u}}_N;\Phi) - a(\tilde{\mathbf{u}}_N,\tilde{\mathbf{u}}_N;\Phi)$$

and

$$s^+(\tilde{\mathbf{u}}_N;\Phi) = s^-(\tilde{\mathbf{u}}_N;\Phi) + \hat{a}(\mathbf{e},\mathbf{e};\Phi),$$

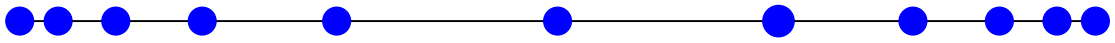
where **e** solves

$$\hat{a}(\mathbf{e},\mathbf{v};\Phi) = l(\mathbf{v};\Phi) - a(\tilde{\mathbf{u}}_N,\mathbf{v};\phi) - b(\mathbf{v},\tilde{p}_N;\Phi) \quad \forall\, \mathbf{v} \in \tilde{X}(\Omega),$$

and

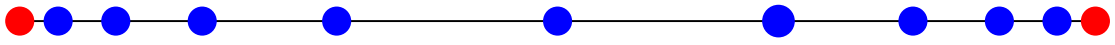
$$\hat{a}(\mathbf{u},\mathbf{v};\Phi) = \int_{\widehat{\omega}} g(\Phi)\widehat{\nabla}(\mathbf{u}\circ\Phi)\cdot\widehat{\nabla}(\mathbf{v}\circ\Phi)d\widehat{\omega}.$$

A basis for the jump discontinuity - tangential direction



Lagrange interpolation polynomials $l_i(\xi) \in \mathbb{P}^{\mathcal{N}}$, defined on $\mathcal{N}+1$ GLL points.

A basis for the jump discontinuity - tangential direction

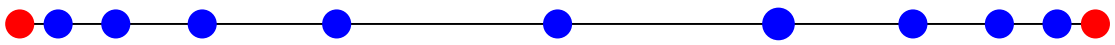


Lagrange interpolation polynomials $l_i(\xi) \in \mathbb{P}^{\mathcal{N}}$, defined on $\mathcal{N}+1$ GLL points.
Find $\hat{\mathbf{u}}_i^t \in X_i^t(\widehat{\Omega})$ and $\hat{p}_i^t \in M(\widehat{\Omega})$, such that

$$
\begin{array}{rcl}
a(\hat{\mathbf{u}}_i^t, \mathbf{v}; \Phi^k) + b(\mathbf{v}, \hat{p}_i^t; \Phi^k) & = & 0 \quad \forall\, \mathbf{v} \in (H_0^1(\widehat{\Omega}))^2 \\
b(\hat{\mathbf{u}}_i^t, q; \Phi^k) & = & 0 \quad \forall\, q \in M_N(\widehat{\Omega}),
\end{array}
$$

where
$X_i^t(\Omega^k) = \{\mathbf{v} \in (H^1(\widehat{\Omega}))^2, \mathbf{v}\cdot\mathbf{n} = 0, \mathbf{v}\cdot\mathbf{t} = l_i(\xi) \text{ on } \widehat{\Gamma}_e, \mathbf{v} = 0 \text{ on } \partial\Omega^k \backslash \Gamma_e\}$.

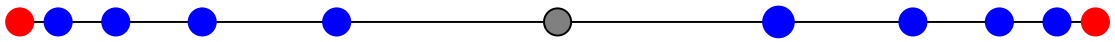
A basis for the jump discontinuity - normal direction



Modified Lagrange interpolation polynomials $\tilde{l}_i(\xi) = l_i(\xi) - q_i(\xi)$, where $q_i(\xi) \in \mathbb{P}^2$ satisfies

$$q_i(-1) = 0, \quad q_i(1) = 0, \text{ and } \int_{-1}^{1} q_i(\xi)d\xi = \int_{-1}^{1} l_i(\xi)d\xi.$$

A basis for the jump discontinuity - normal direction



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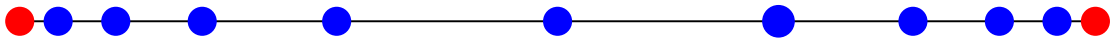
Find $\hat{\mathbf{u}}_i^n \in X_i^n(\widehat{\Omega})$ and $\hat{p}_i^n \in M(\widehat{\Omega})$, such that

$$\begin{array}{lcl} a(\hat{\mathbf{u}}_i^n, \mathbf{v}; \Phi^k) + b(\mathbf{v}, \hat{p}_i^n; \Phi^k) & = & 0 \quad \forall\, \mathbf{v} \in (H_0^1(\widehat{\Omega}))^2 \\ b(\hat{\mathbf{u}}_i^n, q; \Phi^k) & = & 0 \quad \forall\, q \in M_N(\widehat{\Omega}), \end{array}$$

where
$X_i^n(\Omega^k) = \{\mathbf{v} \in (H^1(\widehat{\Omega}))^2, \mathbf{v} \cdot \mathbf{n} = \tilde{l}_i(\xi) \text{ on } \widehat{\Gamma}_e, \mathbf{v} \cdot \mathbf{t} = 0, \mathbf{v} = 0 \text{ on } \partial\Omega^k \backslash \Gamma_e\}$.

A basis for the jump discontinuity - online assembly



The jump across Γ_e

$$[\hat{\mathbf{u}}_e] = \begin{bmatrix} \hat{u}_e^t \\ \hat{u}_e^n \end{bmatrix}$$

The conforming correction is now

$$\hat{\mathbf{u}}_c^k = \sum_{i=1}^{\mathcal{N}-1} \gamma_i^t \hat{\mathbf{u}}_i^t + \sum_{i=1}^{\mathcal{N}-2} \gamma_i^n \hat{\mathbf{u}}_i^n,$$

where the tangential coefficients are

$$\gamma_i^t = \hat{u}_e^t(\xi_i)$$

and the normal coefficients are

$$\begin{bmatrix} \tilde{l}_1(\xi_1) & \dots & \tilde{l}_{\mathcal{N}-2}(\xi_1) \\ \vdots & \ddots & \vdots \\ \tilde{l}_1(\xi_{\mathcal{N}-2}) & \dots & \tilde{l}_{\mathcal{N}-2}(\xi_{\mathcal{N}-2}) \end{bmatrix} \begin{bmatrix} \gamma_1^n \\ \vdots \\ \gamma_{\mathcal{N}-2}^n \end{bmatrix} = \begin{bmatrix} \hat{u}_e^n(\xi_1) \\ \vdots \\ \hat{u}_e^n(\xi_{\mathcal{N}-2}) \end{bmatrix}$$

A basis for the jump discontinuity - online assembly

The jump across Γ_e

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The conforming correction is now

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where the tangential coefficients are

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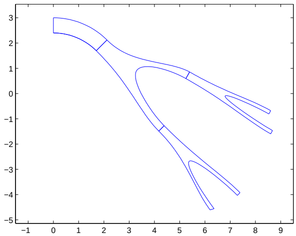
and the normal coefficients are

$$\begin{bmatrix} \tilde{l}_1(\xi_1) & \dots & \tilde{l}_{\mathcal{N}-2}(\xi_1) \\ \vdots & \ddots & \vdots \\ \tilde{l}_1(\xi_{\mathcal{N}-2}) & \dots & \tilde{l}_{\mathcal{N}-2}(\xi_{\mathcal{N}-2}) \end{bmatrix} \begin{bmatrix} \gamma_1^n \\ \vdots \\ \gamma_{\mathcal{N}-2}^n \end{bmatrix} = \begin{bmatrix} \hat{u}_e^n(\xi_1) \\ \vdots \\ \hat{u}_e^n(\xi_{\mathcal{N}-2}) \end{bmatrix}$$

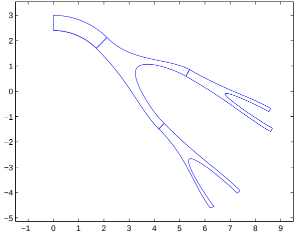
Speedup



Steady Stokes solved with a conventional spectral element code, $\mathcal{N} = 20$

$$t = 4746s \approx 1h20m, \quad s(\mathbf{u}; \Phi) = 4.80 \cdot 10^{-4}$$

Speedup



Steady Stokes solved with a conventional spectral element code, $\mathcal{N} = 20$

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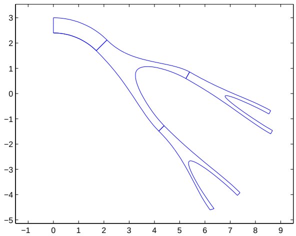
Reduced basis approximation without offline-online decoupling, $N = 15$,

$$t = 299s \approx 5m, \quad s^{+}(\tilde{\mathbf{u}}_N; \Phi) - s^{-}(\tilde{\mathbf{u}}_N; \Phi) = 4.59 \cdot 10^{-6}$$

Speedup



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Reduced basis approximation without offline-online decoupling, $N = 15$,

$$t = 299s \approx 5m, \quad s^{+}(\tilde{\mathbf{u}}_N;\Phi) - s^{-}(\tilde{\mathbf{u}}_N;\Phi) = 4.59 \cdot 10^{-6}$$

Reduced basis approximation with decoupling, $N = 15$,

$$t = 38s, \quad s^{+}(\tilde{\mathbf{u}}_N;\Phi) - s^{-}(\tilde{\mathbf{u}}_N;\Phi) = 4.59 \cdot 10^{-6}$$